Exercise Sheet 2

1. For each of the following functions find the points in which the Cauchy-Riemann equations hold ($z = x + iy$):

   - $f(z) = e^{-x}e^{-iy}$,
   - $f(z) = x^2 + iy^2$,
   - $f(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$,
   - $f(z) = z^2 - \bar{z}^2$.

   Recall that:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

2. Prove that the following function:

$$f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

   satisfies the Cauchy-Riemann equations at $z = 0$, but is not complex differentiable there.

3.
   - Let $f\colon D \to \mathbb{C}$ be holomorphic. Assume that $f(z)$ is real for every $z \in D$. Prove that f is constant.
   - Let $D \subset \mathbb{C}$ be a domain. Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Assume that $\bar{f}$ is also holomorphic. Prove that f is constant.
   - Let $f\colon D \to \mathbb{C}$ be holomorphic. Assume that $|f|$ is constant. Prove that f itself is constant.
   - Let $f\colon D \to \mathbb{C}$ be holomorphic. Assume that there exist real constants $A, B, C \in \mathbb{R}$, A and B not both 0, such that $A\,\mathrm{Re}(f(z)) + B\,\mathrm{Im}(f(z)) + C = 0$, for every $z \in D$, then f is constant.

4. A domain $D \subset \mathbb{C}$ is called symmetric if for every $z \in D$, we have that $\bar{z} \in D$. Let $f\colon D \to \mathbb{C}$ be a holomorphic function with D symmetric. Prove that $g(z) = \overline{f(\bar{z})}$ is also holomorphic.

5. Assume $f\colon D \to \mathbb{C}$ is holomorphic.

   - Let $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. Prove the first polar form of the Cauchy-Riemann equations:
   $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$
   - Let $f(re^{i\theta}) = R(r,\theta)e^{i\phi(r,\theta)} = R(r,\theta)\left(\cos(\phi(r,\theta)) + i\sin(\phi(r,\theta))\right)$. Prove the second polar form of the Cauchy-Riemann equations:
   $$\frac{r}{R}\frac{\partial R}{\partial r} = \frac{\partial \phi}{\partial \theta}, \quad \frac{1}{R}\frac{\partial R}{\partial \theta} = -r\frac{\partial \phi}{\partial r}.$$

6. Let f be holomorphic in $D_R(0)$, for some $R > 0$. Define $g(z) = f\left(\overline{R^2/\bar{z}}\right)$ (the point $R^2/\bar{z}$ is said to be symmetric to z with respect to the circle of radius R around 0). Where is g defined? Prove that g satisfies the Cauchy-Riemann equations. Hint: use polar coordinates.

7. Find the radii of convergence of the following power series:
$$\sum_{n=0}^{\infty} a^{n^2} z^n (0 \neq a \in \mathbb{C}) \ , \ \sum_{n=0}^{\infty} z^{n!} \ , \ \sum_{n=1}^{\infty} n^{-n} z^n.$$

8. Find an example of a power series satisfying each one of the following (a different example for every property):

   - A power series that converge at all points of the boundary of the disc of convergence,
   - A power series that converge at no point of the boundary of the disc of convergence.

   For each of the examples prove the required property.

9. Show that the Weierstrass M-test is not necessary for uniform convergence, i.e., produce an example of a uniformly convergent series (not necessarily power series) such that there exists no suitable dominating series $\sum_{n=0}^{\infty} M_n$.

10. Verify explicitly (using definitions) that given a fixed $0 < \delta < 1$, the power series $\sum_{n=0}^{\infty} z^n$ converges uniformly on $D_{1-\delta}(0)$. Prove that it does not converge uniformly on $D_1(0)$.

11. In this exercise we will prove a light version of a theorem Lucas. Let $p(z)$ be a polynomial. Recall that $z_0 \in \mathbb{C}$ is a root of p if $p(z_0) = 0$.

   - Assume the roots of $p(z)$ are $z_1, \ldots, z_n$. Compute the logarithmic derivative of p, namely $\frac{p'(z)}{p(z)}$ in terms of the roots of p (we will see the logarithmic derivative later in the course, remember it).
   - A half-plane is a domain in $\mathbb{C}$ is a determined by a line $z = a+bt$ for $a, b \in C$ and $t \in \mathbb{R}$. The right half plane is given by $\mathrm{Im}(\frac{z-a}{b}) < 0$ and the left half plane by $\mathrm{Im}(\frac{z-a}{b}) > 0$. Let $H = \{z \in C \mid \mathrm{Im}(\frac{z-a}{b} < 0)\}$, for some $a, b \in \mathbb{C}$. Assume that $u \in H$ and $v \notin H$. Prove that $\mathrm{Im}(\frac{v-u}{b}) > 0$.
   - Using the above two results prove that if the roots of a polynomial p are all in H, the the roots of p' are also in H.

12. Let $f(z) = \frac{p(z)}{q(z)}$ be a rational function. Assume that $|f(z)| = 1$ for every z, such that $|z| = 1$. In particular it implies that there are no zeroes of p or of q on the unit circle. Find the general form of f, in other words describe the relation between the zeroes of p and of q. Hint: define an auxiliary function $g(z) = \frac{1}{f(\frac{1}{\bar{z}})}$. What can you say about the relation between f and g on the unit circle? What does it imply?

13. Let $f(z) = \frac{p(z)}{q(z)}$ be a rational function real on the unit circle ($|z| = 1$). How are the zeroes of p and q situated? Hint: Similarly to the previous exercise define an auxiliary function $g(z) = \overline{f(\frac{1}{\bar{z}})}$.